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Classification of the Surfaces of Singularities of the Quadratic Spherical Complex.

By C. L. E. Moore.

In Vol. 1, page 381, of the Transactions of the American Mathematical Society, Professor P. F. Smith has discussed the surface of singularities of the general quadratic spherical complex. It is the aim of this paper to complete the classification of these surfaces as Weiler* has done for the quadratic line complex. Since spheres as well as lines can be represented by six homogeneous coordinates, Weiler's symbols and notations for the fundamental complexes are used, but no further use is made of line geometry. The properties of those surfaces which can be obtained from cyclides are investigated by means of the transformation used by Smith, and his transformation notation is adopted.

(I).

First Canonical Form. [111111]

1. $$\Pi = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 = 0,$$
$$F = a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2 + a_5x_5^2 + a_6x_6^2 = 0.$$

This is the surface discussed by Smith. He showed that the surface has six double sphero-quartics. The 32 minimum lines are also double, for $(A) = (DID)$ and (D) transforms a minimum line on a surface into two double minimum lines on the transformed surface,† (I) leaves the lines double. Hence the 16 minimum lines of the cyclide by (A) transform into 32 double lines of the transformed surface.

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* Math. Ann., Vol. VII, page 145.

† See Roberts, "On Parallel Surfaces," Proceedings of the London Mathematical Society, Vol. IV, page 233.

These lines are arranged so that each line g is cut by six others, for a minimum line on the cyclide transforms into two intersecting lines on the transformed surface, and two intersecting minimum lines transform into two pairs of intersecting lines. Hence g and the five lines of the cyclide which cut it transform into two intersecting lines and five other lines cutting each one.

Snyder* has shown that the surface of singularities of F is also the surface of singularities for the ∞^1 complexes obtained by giving K all real values in

$$\frac{x_1^2}{k+a_1}+\frac{x_2^2}{k+a_2}+\frac{x_3^2}{k+a_3}+\frac{x_4^2}{k+a_4}+\frac{x_5^2}{k+a_5}+\frac{x_6^2}{k+a_6}=0,$$

and that, consequently, the surface belongs to the congruences

$$x_k=0,\quad \sum_1^6{}_i\,\frac{x_i^2}{a_i-a_k}=0,\qquad i\neq k.$$

2. [(11) 1111].

$$\frac{x_1^2+x_2^2}{k+a_1}+\frac{x_3^2}{k+a_3}+\frac{x_4^2}{k+a_4}+\frac{x_5^2}{k+a_5}+\frac{x_6^2}{k+a_6}=0.$$

In 1, if $x_1=0$ is the complex of points, the surface 111111 becomes the general cyclide.† The generators which belong to $x_2=0$ have double contact with the surface. Hence [(11) 1111] becomes the points of $x_2=0$ which have double contact with the surface, i. e. the focal line (general sphero-quartic, c_4) which lies on the fundamental sphere of $x_2=0$.

By a general transformation (A), the complex $x_1=0$ inverts into a general complex and the focal line c_4 inverts into the surface of singularities of the surface [(11) 1111]. c_4 lies on a non-directed sphere s, therefore, its points invert into spheres which touch the two director spheres into which s inverts; furthermore, c_4 cuts each generator of s in two points which invert into spheres having a line in common with one of the directrices. Hence, *the surface has two double directrices*. It is of order and class 16, passes 8 times through K, the imaginary circle at infinity, the locus of centers is a quartic curve. The lines of curvature‡ are spherical curves of order 16. There are two special ones of order 8 which are the locus of the points of contact of the generators with each of the directrices. From (2) we see that the surface may be generated in four ways as the envelope of ∞^2 spheres, hence it has four developables of bitangent planes, and for the generation as the envelope of ∞^1 spheres, the developable degrades into four planes. c_4 cuts the fundamental sphere of A in four points, hence the transformed surface has four nodes which lie on a circle.

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* Bulletin of the American Mathematical Society, Vol. 4, page 152.

† See Loria, "Ricerche intro alla Geometria Della Sfera," Memorie di Torino, Vol. 36, ser. 2, 1884, page 75.

‡ Snyder, "Lines of Curvature," etc., American Journal, Vol. XXII, page 96.

The complex A may be so chosen that the director spheres may be, (1) two spheres, (2) two points, (3) point and sphere, (4) point and plane, (5) plane and sphere, (6) two planes. In the last case, the lines of curvature and locus of centers are plane curves. The directrices cannot have united position.

If c_4 lies on the sphere s', whose points invert into planes by (A), the surface becomes a developable circumscribed to a sphere along a sphero-quartic. In this case, the special complex which has s' for fundamental sphere, transforms into the complex of planes, i. e. one of the special complexes belonging to the congruence $x_1 + kx_2 = 0$, is the complex of planes, and since x_3, x_4, x_5, x_6 are in involution with it, they are plane complexes (fundamental sphere is a plane). The surface then belongs to that class of surfaces discussed by Smith in the Annals,[*] which we shall call *Laguerre surfaces*.[†] It is the Laguerre transform of the quadric cone as may be seen by transforming x_6 into the complex of points. The fundamental complexes become

$$x_1 + ix_2 = \nu_1\,; \quad x_1 - ix_2 = \mu_1\,; \quad x_3,\, x_4,\, x_5,\, x_6 = \xi,\, \eta,\, \zeta,\, \lambda,$$

and the surface is seen to be a cone.[‡]

By a general (E), the cone becomes the surface described above. It is of order 8, class 4, the characteristics of the cuspidal curve are $m = 12$, $r = 8$, $n = 4$, $d = 0$, etc. The surface does not contain K. Spheres concentric with the director sphere cut the surface in lines of curvature,[§] hence the lines of curvature are of order 16.

The cone has three planes of symmetry, therefore, it is sibireciprocal under three inversions and also λ (the complex of points), hence the transformed surface is sibireciprocal under four inversions, and, consequently, has *four double conics.* This is the complete double curve of the developable.

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[*] Annals of Mathematics, ser. 2, Vol. 1, page 153.

[†] Laguerre surfaces may be defined as the envelope of planes belonging to a quadratic spherical complex. They may be derived from the quadric surfaces by means of an inversion in a plane complex.

[‡] The coordinates ξ, η, ζ, λ, μ, ν are those used by Snyder in "Criteria for Nodes in Dupin's Cyclides," Annals of Mathematics, Vol. 2, June, 1897.

[§] Snyder, "Lines of Curvature on Annular Surfaces." etc., American Journal of Mathematics, Vol. XXII, page 96.

3. $[(11)(11)\,11]$. $a_1 = a_2, \quad a_3 = a_4,$

$$\frac{x_1^2 + x_2^2}{k + a_2} + \frac{x_3^2 + x_4^2}{k + a_3} + \frac{x_5^2}{k + a_5} + \frac{x_6^2}{k + a_6} = 0.$$

In 2, if $x_3 = 0$ is the complex of points, the surface becomes the binodal cyclide, then $[(11)(11)\,11]$ is the focal line on the fundamental sphere of x_4 which, in this case, breaks up into two circles. By a transformation (A), the surface becomes two Dupin cyclides. They have in common two spheres of each generation, viz. the transform of the non-directed sphere on which the circles lie and the transform of their common points. The cyclides are so related that two nodes of one and two nodes of the corresponding generation of the other lie on a circle.

If the sphere on which the circles lie is the sphere whose points transform into planes, the surface becomes two cones of revolution which have two common tangent planes.

4. $[(11)(11)(11)]$.

$$a_1 = a_2, \quad a_3 = a_4, \quad a_5 = a_6.$$

In 3, if $x_1 = 0$ is the complex of points, the focal line on the fundamental sphere of x_2 breaks up into four minimum lines, therefore, the general surface consists of two Dupin cyclides which degrade into two skew quadrilaterals so related that each line of the first cuts one line of the second. These lines can be arranged into two groups of four non-intersecting lines such that any line of one cuts three lines of the other.

For the Laguerre surface, each cyclide becomes two finite minimum lines and* the line at infinity in their plane counted twice.

5. $[(111)\,111]$. $a_1 = a_2 = a_3,$

$$\frac{x_1^2 + x_2^2 + x_3^2}{k + a_1} + \frac{x_4^2}{k + a_4} + \frac{x_5^2}{k + a_5} + \frac{x_6^2}{k + a_6} = 0.$$

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* By finite minimum line is meant a minimum line not lying in plane at infinity.

If $k = -a_1$, we have

$$x_1 = x_2 = x_3 = 0, \quad \frac{x_4^2}{a_4 - a_1} + \frac{x_5^2}{a_5 - a_1} + \frac{x_6}{a_6 - a_1} = 0.$$

Hence the surface belongs to the series $x_1 = x_2 = x_3 = 0$, but the spheres of this series also belong to the series $x_4 = x_5 = x_6 = 0$ and the spheres of the latter series are double, therefore, the surface of singularities consists of a Dupin cyclide counted twice.

If $x_1 = 0$ is the complex of points, the surface of singularities is a circle counted twice. By (*A*) this becomes a Dupin cyclide counted twice.

The Laguerre surface is a cone of revolution counted twice.

The complex [(111) 111] belongs to a series of five, the others being [(111)(11) 1], [(111) 12], [(111)(12)], [(111) 3], which are formed by aid of an involution [2] between the spheres on a Dupin cyclide. We find the four special cases.

1. Two double elements coincide.
2. Three double elements coincide.
3. Four double elements coincide.
4. Two pairs of double elements coincide.

We shall now show that the complex [(111) 111] consists of spheres which touch the corresponding spheres of a Dupin cyclide in an involution [2].

The complex may be written

$$a_4 x_4^2 + a_5 x_5^2 + a_6 x_6^2 = 0.$$

Any sphere of this complex is, therefore, given by

$$\sqrt{a_4}x_4 : \sqrt{a_5}x_5 : \sqrt{a_6}x_6 = \mu^2 - 1 : i(\mu^2 + 1) : 2\mu,$$

which divide the complex into a singly infinite number of linear congruences. Also, any sphere of the cyclide $x_1 = x_2 = x_3 = 0$ is given by

$$x_4 : x_5 : x_6 = \rho^2 - 1 : i(\rho^2 + 1) : 2\rho.$$

All spheres of such a linear congruence touch spheres of the cyclide for values of ρ given by

$$\frac{(\mu^2 - 1)(\rho^2 - 1)}{\sqrt{a_4}} - \frac{(\mu^2 + 1)(\rho^2 + 1)}{\sqrt{a_5}} + \frac{4\mu\rho}{\sqrt{a_6}} = 0.$$

If ρ_1 and ρ_2 are the roots of this quadratic we have

$$\rho_1 + \rho_2 = \frac{A\mu}{C\mu^2 + D}, \quad \rho_1\rho_2 = \frac{D\mu^2 + C}{C\mu^2 + D},$$

where

$$A = \frac{4}{\sqrt{a_6}}, \quad C = \frac{1}{\sqrt{a_5}} - \frac{1}{\sqrt{a_4}}, \quad D = \frac{1}{\sqrt{a_5}} + \frac{1}{\sqrt{a_4}}.$$

Hence, between ρ_1 and ρ_2, the following equation exists:

$$(\rho_1 + \rho_2)^2 = \frac{A^2}{(C^2 - D^2)^2}(C - D\rho_1\rho_2)(C\rho_1\rho_2 - D),$$

which defines the general involution [2].

The complex is, therefore, obtained by establishing a general involution between the spheres of a Dupin cyclide and taking all the spheres which touch each pair of corresponding spheres.

In the case $[(111)(11)1]$ $a_4 = a_5$ and, therefore, $c = 0$ and the involution becomes

$$(\rho_1 + \rho_2)^2 = \frac{A^2}{D^2}\rho_1\rho_2,$$

and, therefore, has two pairs of coincident roots, two zero and two infinite. This is a cubic involution.

6. $[(111)(11)\ 1]$. $\quad a_1 = a_2 = a_3 : a_4 = a_5$.

The surface of singularities in this case belongs to

$$(x_4 + ix_5)(x_4 - ix_5) = x_6 = 0.$$

It is a Dupin cyclide counted twice. The singular spheres of the complex are the spheres common to

$$a_4(x_4^2 + x_5^2) + a_6x_6^2 = 0, \; a_4^2(x_4^2 + x_5^2) + a_6^2x_6^2 = 0,$$

hence they form the congruences

$$x_4 + ix_5 = x_6 = 0 : x_4 - ix_5 = x_6 = 0.$$

The double spheres consist of one generation of the Dupin cyclide together with the two spheres common to

$$x_1 = x_2 = x_3 = x_6 = 0.$$

7. $$[(111)(111)] = a_1 = a_2 = a_3 : a_4 = a_5 = a_6 .$$

Since $\Pi\left(\frac{\partial F}{\partial x_i}\right) = 0$ for all values of x, which satisfy $F = 0$, the complex consists of the tangent pencils of spheres tangent to a Dupin cyclide.*

II.

Second Canonical Form. [11112]

1. $$\Pi = x_1^2 + x_2^2 + x_3^2 + x_4^2 + 2x_5x_6 = 0 ,$$

$$F = a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2 + 2a_5x_5x_6 + x_5^2 = 0 ,$$

$$f = \frac{x_1^2}{k + a_1} + \frac{x_2^2}{k + a_2} + \frac{x_3^2}{k + a_3} + \frac{x_4^2}{k + a_4} + \frac{2x_5x_6}{k + a_5} - \frac{x_5^2}{(k + a_5)^2} = 0 .$$

The surface of singularities is the transform of the nodal cyclide, it is of order and class 20 and passes 10 times through K. It can be generated in five ways as the envelope of ∞^2 spheres, one of which is special, i. e. the linear complex of the congruence is a special complex. For this generation, the developable of bitangent planes becomes extraordinary (simple tangent planes). The transform of the conical point of the cyclide is a double sphere of the transformed surface.

If $x_5 = 0$ is the complex of planes, the surface is the Laguerre surface discussed by Smith.† That it is the Laguerre transform of the central quadric can be seen by making

$$x_1, x_2, x_3, x_4, x_5, x_6 = \xi, \eta, \zeta, \lambda, \mu_1, \nu_1 .$$

Then put $k = - a_4$ and we have

$$\lambda = 0, \frac{\xi^2}{a_1 - a_4} + \frac{\eta^2}{a_2 - a_4} + \frac{\zeta^2}{a_3 - a_4} + \frac{2\mu_1\nu_1}{a_5 - a_4} - \frac{\nu_1^2}{(a_5 - a_4)^2} = 0 .$$

Eliminating $\mu_1\nu_1$ by means of Π and dividing by ν, we have the point equation of a central quadric. By an (E), this quadric transforms into the surface discussed by Smith in the Annals.

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* Snyder, Bulletin of the American Mathematical Soc., Vol. 4, page 146.

† Annals of Mathematics, ser. 2, Vol. 1, page 153. This surface was also discussed by E. Müller, Monatshefte der Math. u. Physik, 1898, Vol. 9, "Die Geometrie orientierter Kugeln nach Grassmannschen Methoden," cf. 7, page 294 ff.

2. [(11) 112]. $a_1 = a_2$.

In 1, if $x_1 = 0$ is the complex of points, the surface is a nodal cyclide, and the focal line on the fundamental sphere of x_2 has a double point. [(11) 112] is then this focal line which, by (*A*), transforms into an annular surface with two double directrices and a double generator. It is of order and class 12, has two general developables of bitangent planes, one extraordinary one and finally one which degenerates into four planes. The locus of centers is a quartic with a double point. The surface can be generated in three ways as the envelope of ∞^2 spheres, one of which is special, and in one way as the envelope of ∞^1 spheres. This surface is also the transform of the central conic since 11112 is a quadric if $x_1 = \lambda$, $x_5 = \nu_1$.

If the focal line lies on the sphere whose points transform into planes, the surface becomes the developable of order 6, circumscribed to a sphere along a nodal sphero-quartic. This is the Laguerre transform of a cone with a minimum tangent plane. The surface has two double conics, its lines of curvature are of order 12.

If $x_5 = 0$ is the complex of planes, the surface is the Laguerre transform of the central quadric of revolution. It is an annular surface of order 8, class 4, has two plane directrices and, consequently, the lines of curvature are plane curves of order 8. It has two finite conics and *K* for double lines. If $x_1 = 0$ is the complex of points, then the surface reduces to a conic. By an (*E*) the latter becomes the same surface as the transform of the quadric of revolution.

It is to be noted here that there are two entirely distinct Laguerre surfaces corresponding to the same symbol [(11) 112]. If the complex of planes is one of the special complexes belonging to the congruence $x_1 = 0$, $x_2 = 0$, the surface is enveloped by ∞^1 planes. By no Laguerre transformation can one be transformed into the other.

3. [111 (12)]. $a_4 = a_5$.

The general nodal cyclide has a focal line on the point sphere x_5. If x_4 be the complex of points, the cyclide reduces to this focal line. In this case, c_4 cuts each generator of the cone in two points, and the spheres on which c_4 lies coincide in the point sphere x_5. By (*A*), c_4 transforms into a surface with coincident double directrices. It is of order and class 16, passes 8 times through *K*. It can be generated in three ways as the envelope of ∞^2 spheres, it has two general developables of bitangent planes and one which degrades. The double curves are three sphero-quartics and the circle of contact with the directrix which is tacnodal.

Since the focal line lies on a point sphere, the developable cannot be generated as in the preceding cases, for (I) would be the only transformation that could be used and this would transform c_4 into a minimum developable. By an (I) with center at x_5, the cyclide $[111\,(12)]$ transforms into a quadric cylinder which by an (E) transforms into a developable of order 8.

In this case the directrices coincide in the plane at infinity. The developable has three finite double conics, and since parallel tangent planes of a cylinder transform into parallel planes, through a line at infinity, which is tangent to the surface, pass two planes of the developable and, consequently, the curve at infinity must be double; as the total double curve is of order 8, that in the plane at infinity must be a conic.

If $x_4 = 0$ is the complex of points and $x_5 = 0$ the complex of planes, the Laguerre surface $[11112]$ is a general quadric. Consider the quadric as generated by its tangent planes, then $[111\,(12)]$ will be the point planes which circumscribe the quadric, i. e. the minimum planes which are tangent to the quadric. Hence, the surface $[111\,(12)]$, when $x_4 = 0$ is complex of points, is the focal developable of the general quadric. It is of order 8, class 4, has three finite conics and K for double lines. This is the minimum developable into which (I) transforms the focal line on the point sphere x_5.

4. $[(11)(11)\,2]$. $\quad a_1 = a_2, \quad a_3 = a_4.$

The cyclide $[(11)\,112)$ has a focal line on x_4 which breaks up into a circle and two minimum lines. Therefore, the surface of singularities of the general complex $[(11)(11)\,2]$ consists of a Dupin cyclide and a skew quadrilateral of minimum lines.

The developable Laguerre surface consists of a cone of revolution and two finite minimum lines and the infinite line in their plane counted twice.

When $x_5 = 0$ is the complex of planes, we obtain immediately from the equations that the Laguerre surface consists of a cone of revolution and a quadrilateral inscribed in K.

5. [1 (11)(12)]. $a_2 = a_3, \quad a_4 = a_5.$

The cyclide [111 (12)] has two tangent circles for focal line on the sphere x_3, therefore, the surface of singularities of the complex [1 (11)(12)] consists of two Dupin cyclides which have two consecutive spheres and, consequently, a circle in common.

When the complex of planes belongs to the congruence $x_2 = 0$, $x_3 = 0$, the Laguerre surface consists of two cones of revolution, tangent along a generator.

When $x_5 = 0$ is the complex of planes, the surface consists of two cylinders of revolution.

If $x_4 = 0$ is complex of points and $x_5 = 0$ complex of planes, the surface becomes the focal developable of the general quadric of revolution. It consists of two minimum cones having their vertices at the foci of a meridian section.

6. [(111) 12]. $a_1 = a_2 = a_3.$

The spheres of the series $x_4 = x_5 = x_6 = 0$ are double for the surface, therefore, the surface of singularities is a Dupin cyclide counted twice.

The Laguerre surface is a cone of revolution counted twice.

The complex may be written

$$a_4 x_4^2 + 2a_5 x_5 x_6 + x_5^2 = 0.$$

It is seen that any sphere of the complex is given by

$$x_4 : x_5 : x_6 = \sqrt{a_5}\,(2\mu a_5 - i) : \sqrt{a_4 a_5} : -\,2\mu\sqrt{a_4}\,(\mu\sqrt{a_5} - i),$$

giving ∞^1 linear congruences; the sphere $(0, 0, 0, 2\rho, 2, \rho^2)$ is any sphere of the cyclide $x_1 = x_2 = x_3 = 0$ and is touched by the spheres of the preceding congruence, provided that

$$\sqrt{a_4 a_5}\,\rho^2 + 4\sqrt{a_4}\,\mu\,(\mu\sqrt{a_5} - i) - 2\rho\sqrt{a_5}\,(2\mu\sqrt{a_5} - i) = 0.$$

This equation defines an involution [2], which has two coincident double elements.

7. [(111)(12)]. $a_1 = a_2 = a_3, \; a_4 = a_5.$

In this case the involution is defined by

$$(\rho_1 - \rho_2)^2 + \frac{a_4}{4} = 0,$$

which has all its double elements coincident.

8. [11(112)]. $a_3 = a_4 = a_5$,

$$F = a_1x_1^2 + a_2x_2^2 + x_5^2 = 0.$$

The singular spheres are those which belong to $F = 0$ and satisfy

$$a_1^2x_1^2 + a_2^2x_2^2 = 0.$$

Thus the singular surface consists of the Dupin cyclide $x_1 = x_2 = x_5 = 0$, which degrades into a skew quadrilateral of minimum lines. The Laguerre surface consists of a quadrilateral inscribed in K. The complex is composed of the ∞^1 congruences

$$x_1 + ix_2 = 2\rho x_5,\ x_1 - ix_2 = 2\sigma x_5,$$

where ρ and σ are connected by the relation

$$a_1(\rho + \sigma)^2 - a_1(\rho - \sigma)^2 + 1 = 0.$$

Hence the complex is composed of spheres which touch corresponding spheres of two pencils which are in (2, 2) correspondence. The correspondence has two of its four double elements in coincidence

9. [(11)(112)]. $a_1 = a_2$, $a_3 = a_4 = a_5$.

This complex consists of the singly infinite number of congruences

$$2\sqrt{a_1}\,(x_1 + ix_2) = \mu x_5,\quad 2\sqrt{a_1}\,(x_1 - ix_2) = -\frac{1}{\mu}x_5.$$

The directrices of these congruences form a tangent pencil of spheres and are in (1, 1) correspondence. Therefore, the complex consists of those spheres which touch corresponding spheres of two projective tangent pencils. The surface of singularities is same as [11 (112)].

III.

Third Canonical Form.

1. [1113]

$$F = a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4(x_5^2 + 2x_4x_6) + 2x_4x_5 = 0,$$

$$\Pi = x_1^2 + x_2^2 + x_3^2 + x_5^2 + 2x_4x_6 = 0,$$

$$f = \frac{x_1^2}{k + a_1} + \frac{x_2^2}{k + a_2} + \frac{x_3^2}{k + a_3} + \frac{x_5^2 + 2x_4x_6}{k + a_4} - \frac{2x_4x_5}{(k + a_4)^2} + \frac{x_4}{(k + a_4)^3} = 0.$$

The surface [1113] is the transform of the cyclide with a general biplanar point. The biplanar point transforms into a cuspidal sphere, as may be seen by considering the transform of every plane section through this point. They will all have the transform of this point for cuspidal sphere and will envelope the surface. It is of order and class 18, can be generated in four ways as the envelope of a doubly infinite number of spheres, one of which is special. The surface is the envelope of those spheres of the complex which touch a fixed sphere of the complex; it has three general developables of bitangent planes and one extraordinary.

If $x_4 = 0$ is the complex of planes, the surface is the Laguerre transform of the paraboloid, as can be seen by transforming one of the fundamental complexes $x_1 = 0$, say, into the complex of points. Then the surface becomes the envelope of

$$\frac{x_2^2}{a - a} + \frac{x_3^2}{a - a} + \frac{2x_4x_5}{(a - a)^2} + \frac{x_4^2}{(a - a)^3} = 0, \quad x_1 = 0.$$

Or, using ξ, η, ζ, λ, μ, ν, the points of the complex envelope, the paraboloid

$$\frac{x^2}{a_2 - a} + \frac{y^2}{a_3 - a_1} - \frac{2z}{(a_4 - a_1)^2} + \frac{1}{(a_4 - a_1)^3} = 0.$$

Since the paraboloid has two planes of symmetry, the surface is sibireciprocal under three inversions and, therefore, has three double conics. Since the paraboloid is tangent to the plane at infinity, the transformed surface will have K for simple line. It is of order 10, class 4.

2. [(11)13]. $a_1 = a_2$.

The cyclide [1113] has for focal line on the fundamental sphere of $x_2 - 0$, a quartic c_4 with a cusp, then the general surface [(11) 13] will be an annular surface which has two double spherical directrices and a cuspidal generator. It is of order and class 10, can be generated in two ways as the envelope of ∞^2 spheres, one of which is special and in one way as the envelope of ∞^1 spheres, it has one general developable of bitangent planes, one extraordinary, and one which degrades. The locus of centers is a skew quartic with a cusp.

If the quartic lies on the sphere whose points transform into planes, the surface becomes the developable of order 5, touching a sphere along a sphero-quartic with a cusp. It has one double conic. The cuspidal edge is a quartic having a cusp.

If $x_4 = 0$ is the complex of planes, the surface is the Laguerre transform of the paraboloid of revolution or of the parabola. It is of order 4, class 5, contains one double conic.

3. [11 (13)]. $a_3 = a_4$.

The cyclide [1113] has a focal line on the point sphere x_4 (minimum cone having x_4 for vertex), which has a double point at the vertex. Each generator of the cone cuts the quartic in but one point aside from the vertex. Then the general surface [11 (13)] will be an annular surface with a single directrix, which is also a double generator. It is of order and class 12.

If $x_4 = 0$ is the complex of planes, the surface is the Laguerre transform of the parabolic cylinder. It is of order 6, class 4, has two doube conics. The director sphere coincides with the plane at infinity.

If $x_4 = 0$ is the complex of planes and $x_3 = 0$ complex of points, the surface is the focal developable of the paraboloid. It is of order 6, class 4, has two finite double conics; K is simple line on the surface.

4. [1 (113)].

$$a_2 = a_3 = a_4\,,$$
$$F = a_1x_1^2 + 2x_4x_5 = 0\,.$$

The singular spheres satisfy $F = 0$ and

$$a_1^2x_1^2 + x_4^2 = 0\,.$$

Thus the surface of singularities consists of the Dupin cyclide $x_1 = x_4 = x_5 = 0$, which degrades into a skew quadrilateral of minimum lines.

The spheres of the complex belong to the single infinite number of linear congruences,

$$a_1(x_5 + ix_1) + 2\mu x_4(a_1\mu - i) = 0\,,$$
$$a_1(x_5 - ix_1) + 2\mu x_4(a_1\mu + i) = 0\,.$$

The directrices of these congruences form two pencils in (2, 2) correspondence. Thus the complex is made up of spheres which touch corresponding spheres of two tangent pencils in (2, 2) correspondence.

5. [(11)(13)]. $a_1 = a_2, \quad a_3 = a_4.$

The focal line of the cyclide [11 (13)] on the sphere x_2 breaks up into a circle and two minimum lines which intersect on it, therefore, the general surface [(11)(13)] consists of a Dupin cyclide and skew quadrilateral of minimum lines lying on one of its generating spheres.

The Laguerre surface consists of a cone of revolution and two minimum lines lying in one of its generating planes.

If $x_3 = 0$ is the complex of points, the surface is the focal developable of the paraboloid of revolution. It is a minimum cone with focus as vertex.

6. [(111) 3] $a_1 = a_2 = a_3$

The surface of singularities is a Dupin cyclide counted twice.

The singular spheres form the special congruences $x_4 = x_5 = 0$, together with a general linear congruence.

The spheres of the complex are given by the equations

$$x_4 : x_5 : x_6 = a_4^{\frac{3}{2}} : 2\mu a_4 : - 2\mu\,(1 - \mu\sqrt{a_4}),$$

forming ∞^1 linear congruences; the spheres of such a congruence touch the sphere $(0, 0, 0, 2\rho^2, 2\rho, - 1)$ of the cyclide $x_1 = x_2 = x_3 = 0$, provided that

$$4\mu\rho^2\,(1 + \mu\sqrt{a_4}) - 4\mu a_4\rho + a_4^{\frac{3}{2}} = 0.$$

If the roots of this equation are ρ_1, ρ_2, it is seen that

$$(\rho_1 - \rho_2)^2 + \frac{4}{a_4}\,\rho_1\rho_2\,(\rho_1 + \rho_2) = 0$$

this is an involution [2], in which three double elements coincide. The Laguerre surface is a cone of revolution counted twice.

IV.

Fourth Canonical Form. [1122]

1. $F = a_1x_1^2 + a_2x_2^2 + 2a_3x_3x_4 + 2a_4x_5x_6 + x_3^2 + x_5^2 = 0,$

$\Pi = x_1^2 + x_2^2 + 2x_3x_4 + 2x_5x_6 = 0,$

$$f = \frac{x_1^2}{k + a_1} + \frac{x_2^2}{k + a_2} + \frac{2x_3x_4}{k + a_3} - \frac{x_3^2}{(k + a_3)^2} + \frac{2x_5x_6}{k + a_4} - \frac{x_5^2}{(k + a_4)^2} = 0.$$

The surface of singularities is the transform of the non-annular binodal cyclide, therefore, it has two double spheres which touch each other. It is of order and class 16, can be generated in four ways as the envelope of ∞^2 spheres, two of which are special; it has two general developables of bitangent planes and two extraordinary.

If $x_1 = 0$ is the complex of points and $x_3 = 0$ the complex of planes, the surface reduces to a quadric, which is tangent to K at one point, then by a Laguerre transformation this quadric inverts into a surface having a double plane. It is of order 10, class 4, has K for double line and has two finite double conics.

2. [(12) 12]. $a_1 = a_3$.

The cyclide [1212] has on the point sphere x_3 a focal line which has a double point, not at the vertex of the minimum cone. Therefore, the general surface is an annular surface which has coincident directrices and a double generator. It is of order and class 12, can be generated in two ways as the envelope of ∞^2 spheres, one of which is special; there are two developables of bitangent planes, one of which is extraordinary. The developable corresponding to the generation as the envelope of ∞^1 spheres degrades into planes.

If $x_3 = 0$ is the complex of planes and $x_2 = 0$ the complex of points, the surface is a quadric cylinder which is tangent to K in one point. By a Laguerre transformation this inverts into a developable of order 6, class 4, has one finite double conic and one in the plane at infinity.

If $x_1 = 0$ is complex of points the surface is the focal developable of a quadric which touches K. It is of order 6, class 4, has one finite conic and K for double curve.

If $x_5 = 0$ is the complex of planes and $x_2 = 0$ the complex of points, the surface becomes a quadric of revolution such that the two points of contact with K coincide. By a Laguerre transformation this inverts into an annular surface with coincident plane directrices, it is of order 8, class 4, contains K as double line and has one finite double conic.

3. [(12)(12)]. $a_1 = a_2,\ a_3 = a_4$.

The cyclide [12 (12)] has a focal line on the point sphere x_3 consisting of two tangent circles. Therefore, the surface of singularities of the general complex [(12)(12)] consists of two Dupin cyclides which touch along a circle (since they have consecutive generating spheres in common), and as the two directrices coincide, they must intersect along a circle of the second system, i. e. the cyclides have a circle of each system in common.

If $x_3 = 0$ is the complex of planes, the surface is given by

$$\frac{x_1^2 + 2x_3x_4}{a_1} + \frac{x_3^2}{a_1^2} + x_6^2 = 0 \qquad x_2 = 0,\ x_5 = 0,$$

which is two lines lying in a minimum plane. By a Laguerre transformation, these transform into two cones of revolution having a generating sphere and *one* generating plane in common.

4. [(11) 22]. $a_1 = a_2$.

The cyclide [1122] has a focal line on the sphere x_2 which degrades into a cubic and a double secant. Therefore, the general surface has a single and a double directrix. It consists of an annular surface of order and class 8, and two minimum lines, one on each directrix, which belong to two non-consecutive generators of the annular surface.

If the sphere x_2 is the sphere whose points transform into planes, the cubic inverts into a developable of order 4, class 3, and the line inverts into a minimum line which lies on the directrix sphere of the developable.

If $x_1 = 0$ is the complex of points and $x_3 = 0$ the complex of planes, the surface becomes a conic passing through one of the circle points. By a Laguerre transformation this becomes an annular surface of order 6, class 4.

5. [11 (22)]. $a_3 = a_4$.

The cyclide [11 (22)] is ruled and has a double minimum line. By a transformation (A), the minimum line d transforms into two lines d', d'', and a line cutting d transforms into two lines, one cutting d' and one cutting d'', and the pair of lines, cutting d in the same point, transform into two pairs of lines, one pair intersecting on each line d' and d''. Two lines intersecting on K invert into two pairs of lines, each intersecting on K and one line of the pair cuts d' and the other d'', therefore, there is a (2, 2) correspondence between d' and K and between d'' and K. Hence, the surface of the singularities consist of two ruled cyclides.

If $x_3 = 0$ is the complex of planes and $x_1 = 0$ the complex of points, the surface is a cone with vertex on K, and by an (E), it transforms into two such cones.

The equation of the complex is

$$a_1x_1^2 + a_2x_2^2 + x_3^2 + x_5^2 = 0,$$

and, therefore, can be broken up into the congruences

$$-\mu\sqrt{a_1a_2}x_1 - i\mu\sqrt{a_1a_2}x_2 + (\mu^2\sqrt{a_1}(\sqrt{a_1} + \sqrt{a_2}) + \sqrt{a_1} - \sqrt{a_2})x_3$$
$$+ i(\mu^2\sqrt{a_1}\sqrt{a_1} + \sqrt{a_2}) - \sqrt{a_1} + \sqrt{a_2})x_3 = 0,$$
$$\mu\sqrt{a_1a_2}x_1 + i\mu\sqrt{a_1a_2}x_2 + (\mu^2\sqrt{a_1}(\sqrt{a_1} - \sqrt{a_2}) + \sqrt{a_1} + \sqrt{a_2})x_3$$
$$+ i(\mu^2\sqrt{a_1}(\sqrt{a_1} - \sqrt{a_2}) - \sqrt{a_1} - \sqrt{a_2})x_5 = 0.$$

The directrices of these congruences form two ruled cyclides which are projective with each other. Therefore, the complex consists of those spheres which touch corresponding spheres of two ruled cyclides which are projective with each other.

6. [(112) 2]. $a_1 = a_2 = a_3$.

The surface of singularities belongs to the series $x_3 = x_4 = x_5 = 0$, and, therefore, consists of a skew quadrilateral of minimum lines counted twice.

If $x_5 = 0$ is complex of planes, the Laguerre surface is a quadrilateral inscribed in K. If $x_3 = 0$ is complex of planes, the Laguerre surface is two finite minimum lines which intersect and the line at infinity in their plane counted twice.

The complex is formed of spheres which belong to the congruence

$$\mu x_3 + x_5 = 0, \quad 2\mu a_3x_4 - (\mu^2 + 1)x_5 = 0,$$

and is, therefore, formed of spheres which touch paired spheres of two pencils in (1, 2) correspondence, which have a common self-corresponding element.

7. [1 (122)]. $a_2 = a_3 = a_4$.

The singular spheres belong to the series $x_1 = x_3 = x_5 = 0$, and the surface is two minimum lines, each counted four times.

If $x_3 = 0$ is the complex of planes, the surface becomes a finite minimum line and a line in the plane at infinity cutting it, each counted four times.

8. [(11)(22)]. $a_1 = a_2,\ a_3 = a_4$.

In the cyclide [11 (22)] there is a focal line on the sphere x_3 consisting of two skew lines and one of their secants counted twice. Therefore, the surface of singularities of the general complex [(11)(22)] consists of two skew quadrilaterals of minimum lines having two common sides. The complex is formed of spheres which belong to the complexes

$$a\mu\,(x_1 + ix_2) + x_3 + ix_5 = 0,$$
$$a_1\,(x_1 - ix_2) - \mu\,(x_3 - ix_5) = 0.$$

The directrices of these congruences form two projective tangent pencils of spheres so related that one of the minimum lines enveloped by the first pencil cuts one of the minimum lines enveloped by the second pencil.

V.

Fifth Canonical Form. [114]

1. $F_1 = a_1x_1^2 + a_2x_2^2 + 2a_3\,(x_3x_6 + x_4x_5) + 2x_3x_5 + x_4^2 = 0,$

$$\Pi = x_1^2 + x_2^2 + 2x_3x_6 + 2x_4x_5 = 0,$$

$$f = \frac{x_1^2}{k+a_1} + \frac{x_2^2}{k+a_2} + \frac{2x_3x_6 + 2x_4x_5}{k+a_3} - \frac{2x_3x_5 + x_4^2}{(k+a_3)^2} + \frac{2x_3x_4}{(k+a_3)^3} - \frac{x_3^2}{(k+a_3)^4} = 0.$$

The surface of singularities is the transform of the cyclide with a special biplanar point. It is, therefore, of order and class 16. The biplanar point inverts into a sphere analogous to a tacnode, as we shall see in the case [(11) 4]. The surface can be generated in three ways as the envelope of ∞^2 spheres, one of which is special; it has two general developables of bitangent planes and one extraordinary. The surface is the envelope of spheres which touch a singular sphere of a quadratic complex (obtained by putting $k = -a_3$).

If $x_1 = 0$ is the complex of points and $x_3 = 0$ the complex of planes, the surface reduces to a paraboloid, which is tangent to the plane at infinity at a point of K. By a Laguerre transformation, this inverts into a surface of order 9, class 4, containing K as simple line; it has two double conics.

2. [(11) 4]. $a_1 = a_4$.

The cyclide [114] has a focal line on the sphere x_2 consisting of a twisted cubic and a minimum line tangent to it. Therefore, the general surface consists of an annular surface of order 8, and two minimum lines lying on two consecutive generating spheres. The surface can be generated in one way as the envelope of ∞^2 spheres; it has no general developable of bitangent planes, but has one extraordinary and one which degrades.

If the cubic and tangent lie on the sphere whose points transform into planes, the surface becomes a developable of order 4 and a minimum line.

If $x_3 = 0$ is the complex of planes and $x_1 = 0$ is the complex of points, the surface becomes a parabola tangent to the line at infinity in one of the circle points. By a Laguerre transformation this inverts into a surface of order 5, containing K as simple line.

3. [1 (14)]. $a_2 = a_3$.

The cyclide [114] has on the point sphere x_3 a focal line having a cusp at the vertex of the minimum cone. Therefore, the general surface [1 (14)] is an annular surface with a single directrix which is also a cuspidal generator, it is of order and class 10.

If $x_3 = 0$ is the complex of planes and $x_1 = 0$ the complex of points, the surface becomes a parabolic cylinder whose generator in the plane at infinity is tangent to K. By a Laguerre transformation this becomes a developable of order 5, class 4, having one double conic. The cuspidal edge is a quartic having a cusp.

If $x_2 = 0$ is the complex of points, the surface is the focal developable of a paraboloid which is tangent to the plane at infinity in a point of K. It is of order 5, class 4, has one double conic and contains K as simple line.

4. [(114)]. $a_1 = a_2 = a_3$.

The surface of singularities belongs to $x_3 = x_4 = x_5 = 0$ and, therefore, consists of a skew quadrilateral of minimum lines counted twice.

If $x_3 = 0$ is the complex of planes, the surface is a quadrilateral inscribed in K.

The complex consists of the singly infinite number of congruences

$$2x_3 - \mu x_4 = 0, \quad 2x_3 - \mu^2 x_5 = 0.$$

The directrices of these special congruences form two tangent pencils in (2, 1) correspondence having a self-corresponding element. The involution formed by the pair of elements of the first pencil corresponding to the elements of the second pencil, has the common self-corresponding element for double element. The complex is, therefore, composed of those spheres which touch corresponding spheres of two tangent pencils in (2, 1) correspondence having a self-corresponding element.

VI.

Sixth Canonical Form. [123]

1. $$F = a_1x_1^2 + 2a_2x_2x_3 + x_2^2 + a_3(2x_4x_6 + x_5^2) + 2x_4x_5 = 0,$$
$$\Pi = x_1^2 + 2x_2x_3 + 2x_4x_6 + x_5^2 = 0,$$
$$f = \frac{x_1^2}{k+a_1} + \frac{2x_2x_3}{k+a_2} - \frac{x_2^2}{(k+a_2)^2} + \frac{2x_4x_6 + x_5^2}{k+a_3} - \frac{2x_4x_5}{(k+a_3)^2} + \frac{x_4^2}{(k+a_3)^3} = 0.$$

The surface is the transform of the cyclide, which has a conical and a general biplanar point; it, therefore, has a double and cuspidal sphere which touch. It is of order and class 14, can be generated in three ways as the envelope of ∞^2 spheres, two of which are special, and has one general developable of bitangent planes and two extraordinary ones.

If $x_2 = 0$ is the complex of planes and $x_1 = 0$ the complex of points, the surface becomes a quadric which has three points contact with K. By a Laguerre transformation, this becomes a surface of order 9, class 4, has a cuspidal plane, has K and a finite conic for double curves.

If $x_4 = 0$ is the complex of planes the surface becomes a parabaloid tangent to K. By a Laguerre transformation, this becomes a surface of order 8, class 4, containing K as simple line. It has one double conic and one double plane.

2. [(12) 3]. $a_1 = a_2$.

The cyclide [123] has a focal line on the point sphere x_2 which has a cusp not at the vertex of the minimum cone. Therefore, the general surface [(12) 3] is an annular surface which has coincident directrices and a cuspidal generator. It is of order and class 10.

If $x_4 = 0$ is the complex of planes, and $x_1 = 0$ the complex of points, the surface becomes a general parabola lying in a minimum plane; by a Laguerre transformation this becomes an annular surface with coincident plane directrices. It is of order 6, class 4, and contains K as simple line.

If x_2 is on the fundamental sphere of (A) the surface becomes one with a five-fold point at x_2. By a transformation by reciprocal radii, this inverts into a developable of order 5 (the Laguerre surface if $x_2 = 0$ is complex of planes).

3. [(13) 2]. $\qquad a_1 = a_3$.

The cyclide [123] has a focal line on x_4 consisting of a cubic and a secant. The general surface is the same as [(11) 22] except that the directrices coincide.

If $x_4 = 0$ is the complex of planes, the developable is of order 4.

If $x_1 = 0$ is the complex of points, the developable becomes the focal developable of a paraboloid tangent to K. It is of order 4, class 3, has K for simple line.

If $x_2 = 0$ is the complex of planes, the annular surface is of order 6. For if x_2 is on the fundamental sphere of (A), the transformed surface will have in x_2 a four-fold point arising from the union of two conical points, but the minimum line goes through this point, therefore, the annular surface has this point for conical point, and by an (I) with centre at x_2, the surface inverts into an annular surface of order 6.

4. [1 (23)]. $\qquad a_2 = a_3$.

The cyclide [1 (23)] is ruled, and for same reason as [11 (22)], the general surface consists of two such cyclides. The double line in this cyclide is simple directrix and simple generator.

The Laguerre surface, as in [11 (22)], consists of two parabolic cylinders with vertex on K.

5. [(123)]. $\qquad a_1 = a_2 = a_3$,

$$F = x_2 - 2x_4x_5 = 0,$$

$$\Pi\left(\frac{\partial F}{\partial x_i}\right) = x_4^2 = 0.$$

The surface of singularities belongs to the series $x_2 = x_4 = x_5 = 0$ and since Δ*

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* See Snyder, "Criteria for Nodes," etc., Annals of Math., 1897.

and its first minors vanish, the quadrilateral becomes two intersecting minimum lines counted four times.

If $x_2 = 0$ or $x_4 = 0$ is the complex of planes, the surface is a line at infinity counted four times.

VII.

Seventh Canonical Form. [15]

1. $$\Pi = x_1^2 + 2x_2x_6 + 2x_3x_5 + x_4^2 = 0,$$
$$F = a_1x_1^2 + a_2(2x_2x_6 + 2x_3x_5 + x_4^2) + 2x_2x_5 + 2x_3x_4 = 0,$$
$$f = \frac{x_1^2}{k+a_1} + \frac{2x_2x_6 + 2x_3x_5 + x_4^2}{k+a_2} - \frac{2x_2x_5 + 2x_3x_4}{(k+a_2)^2} + \frac{2x_2x_4 + x_3^2}{(k+a_2)^3} - \frac{2x_2x_3}{(k+a_2)^4} + \frac{x_2^2}{(k+a_2)^5} = 0.$$

The surface of singularities is the inverse of the cyclide, which has a special biplanar point. The surface is of order and class 14; it can be generated in two ways as the envelope of ∞^2 spheres, one of which is special, therefore, it has one general developable of bitangent planes and one extraordinary one.

If $x_1 = 0$ is complex of points and $x_2 = 0$ the complex of planes, the surface becomes a paraboloid, which is tangent to the plane at infinity in a point of K, and one of the lines in the plane at infinity is tangent to K. By a Laguerre transformation, this transforms into a surface of order 8, class 4, having one double conic and having K for simple line.

2. [(15)]. $a_1 = a_2$.

The cyclide [15] has a focal line on the point sphere x_2 consisting of a twisted cubic and a minimum line tangent to it. The general surface consists of an annular surface as in [(11) 4] except that the directrices coincide.

The Laguerre surface is a developable of order 4, and a minimum line. This differs from [(21) 4] in having the director sphere coincide with the plane at infinity.

If $x_1 = 0$ is the complex of points and $x_2 = 0$ the complex of planes, the Laguerre surface becomes the focal developable circumscribed to the paraboloid [15]. It is of order 4, contains K as simple line.

VIII

Eighth Canonical Form. [222]

1. $$F = 2a_1x_1x_2 + 2a_2x_3x_4 + 2a_3x_5x_6 + x_1^2 + x_3^2 + x_5^2 = 0,$$
$$\Pi = x_1x_2 + x_3x_4 + x_5x_6 = 0.$$

If $x_5 = 0$ is the complex of planes the surface is the Laguerre surface having two double planes, and as the Laguerre surfaces are of class 4, two double planes will reduce the order of the surface to 8. The minimum line common to $x_1 = x_3 = x_5 = 0$ is also a part of the surface, since these spheres are all singular spheres. The presence of this line does not reduce the order nor class of the residual surface.

By an (I), this inverts into the general surface of singularities, which is, therefore, of order and class 12, passes six times through K. The minimum line enveloped by $x_1 = x_3 = x_5 = 0$ is also a part of this surface. The surface can be generated in three ways as the envelope of ∞^2 spheres, two of which are special. It has three double spheres which have a minimum line in common.

2. [2 (22)]. $a_2 = a_3$,
$$F = 2a_1x_1x_2 + x_1^2 + x_3^2 + x_5^2 = 0,$$
$$f = \frac{2x_1x_2}{a_1 - a_2} - \frac{x_1^2}{(a_1 - a_2)^2} + x_4^2 + x_6^2 = 0, \quad x_3 = 0, \quad x_5 = 0.$$

Since $x_3 = 0$, $x_5 = 0$, we have at once from Π either $x_1 = 0$ or $x_2 = 0$. In the first case, spheres which belong to $x_1 = x_3 = x_5 = 0$ and f, from two tangent pencils of spheres which have a sphere in common. In the second case, spheres which belong to $x_2 = x_3 = x_5 = 0$ and f, form a ruled cyclide. Hence, the surface of singularities consists of two tangent pencils of spheres and a ruled cyclide.

The complex may be broken up into the congruences

$$x_1 - \mu x_3 - i\mu x_5 = 0,$$
$$2a_1\mu x_2 + (1 + \mu^2) x_3 - i(1 - \mu^2) x_5 = 0,$$

the directrices of these congruences are

$$S' \equiv (0, 1, 0, -\mu, 0, -i\mu),$$
$$S \equiv (2a\mu, 0, 0, 1 + \mu^2, 0, -i(1 - \mu^2)).$$

The spheres S' form a tangent pencil. The spheres S form a ruled cyclide. Therefore, the complex consists of spheres which touch paired spheres (S', S).

The pencils and the cyclide have a self-corresponding sphere. If the sign of i be changed, i. e. if the congruences be found in another way, we get the same ruled cyclide but a different pencil. The two pencils have a sphere in common thus the complex can be generated in two ways as above.

3. [(222)].

$$a_1 = a_2 = a_3.$$
$$F = x_1^2 + x_3^2 + x_5^2 = 0,$$
$$\Pi\left(\frac{\partial F}{\partial x_i}\right) = 0.$$

Every sphere of the complex is singular, and as x_1, x_3, x_5 are special and in involution, the surface is generated by spheres which have a minimum line in common and which satisfy a quadratic complex. This is the ruled cyclide. For, if S_1, S_3, S_5 be three such spheres, then any sphere can be represented by

$$S_\iota = x_2 s_1 + x_4 s_3 + x_6 s_5 = 0,$$
$$\pi = x_1 x_2 + x_3 x_4 + x_5 x_6 = 0.$$

The spheres S_i belong to $x_1 = x_3 = x_5 = 0$ and, therefore, the envelope subject to the quadratic relation is

$$a_1' a_2' s_1^2 + a_1' s_3^2 + a_2' s_5^2 = 0,$$

which is a ruled cyclide, since s_1, s_3, s_5 have a minimum line in common.

IX.

Ninth Canonical Form. [33]

1.
$$F = a_1(2x_1x_3 + x_2^2) + 2x_1x_2 + a_2(2x_4x_6 + x_5^2) + 2x_4x_5 = 0,$$
$$\Pi = 2x_1x_3 + x_2^2 + 2x_4x_6 + x_5^2 = 0,$$
$$f = \frac{2x_1x_3 + x_2^2}{k + a} - \frac{2x_1x_2}{(k + a_1)^2} + \frac{x_1^2}{(k + a_1)^3} + \frac{2x_4x_6 + x_5^2}{k + a_2}$$
$$- \frac{2x_4x_5}{(k + a_2)^2} + \frac{x_4^2}{(k + a_2)^3} = 0.$$

If $x_4 = 0$ is the complex of planes, the surface is a Laguerre surface having a cuspidal plane. The surface [1113] is of order 10, and as a cuspidal plane reduces the order of the surface three, the order of the surface [33] is 7.

By an (I), this inverts into the general surface of singularities, therefore, the general surface of singularities is of order and class 12, passes six times through K. It has two cuspidal spheres which touch each other.

2. [(33)].
$$a_1 = a_2,$$
$$F = x_1 x_2 + x_4 x_5 = 0.$$

The complex may be decomposed into the congruences

$$x_1 + x_4\, 2\mu x_2' = 0, \quad x_1 - x_4 - 2i\mu x_5' = 0, \quad [x_2 = x_2' + ix_5',\ x_5' = x_2' - ix_5'].$$

The directrices of the congruences are

$$S_1 \equiv (0,\ 0,\ 1,\ 0,\ 2\mu,\ 1),$$
$$S_2 \equiv (0,\ -2i\mu,\ 1,\ 0,\ 0,\ -1),$$

and form two tangent pencils of spheres in (1, 1) correspondence such that the sphere $T_1 \equiv (0, 0, 1, 0, 0, -1)$ touch all the spheres S_1, and the sphere $T \equiv (0, 0, 1, 0, 0, 1)$ touch all the spheres S_2.

If the congruences were

$$x_1 + \mu x_4 = 0, \quad \mu x_2 - x_5 = 0,$$

the directrices all coincide in the pencil. $S \equiv (0, 0, 1, 0, 0, \mu)$. The two minimum lines enveloped by this pencil are such one belongs to S_1 and the other to S_2. Therefore, in the surface of singularities, these lines count as triple line and the other two as simple lines.

X.

Tenth Canonical Form. [24]

1.
$$F = 2a_1 x_1 x_2 + x_1^2 + a_2 (2x_3 x_6 + 2x_4 x_5) + 2x_3 x_5 + x_4^2 = 0,$$
$$\Pi = x_1 x_2 + x_3 x_6 + x_4 x_5 = 0.$$

In this case there are two Laguerre surfaces according as x_1 or x_3 is the complex of planes. The first is a surface with a tacnodal plane, and, therefore, of order 8, it has K for double line. The second differs from [114] by having a double plane and, therefore, is of order 7, and has K for simple line.

By an (I), these invert into the general surface which is, therefore, of order and class 12; has one double and one tacnodal sphere. In this case, as in the case [222], a minimum line forms part of the surface.

2. [(24)].
$$a_1 = a_2,$$
$$F = x_1^2 + 2x_3 x_5 + x_4^2 = 0,$$
$$f = x_2^2 + 2x_4 x_6 + x_5^2 = 0, \quad x_1 = 0, \quad x_3 = 0.$$

Either $x_4 = 0$ or $x_5 = 0$. In the first case spheres common to $x_1 = x_3 = x^4$ and f envelope two tangent pencils; in the second case, spheres common to $x_1 = x_3 = x_5$ and f form a ruled cyclide.

The complex consists of the congruences

$$2\mu x_3 - x_1 - ix_4 = 0, \quad 2\mu x_1 - 2\mu^2 x_3 + x_5 = 0.$$

The directrices of these special congruences are

$$S_1 \equiv (0, -1, 0, 0, -i, 2\mu),$$
$$S_2 \equiv (0, 2\mu, 0, 1, 0, -2\mu^2).$$

The spheres S_1 form a tangent pencil and the spheres S_2 form a ruled cyclide. S_1 has one sphere in common with S_2, viz. $(0, 0, 0, 0, 0, 2)$ and the sphere of S_1 touch no other sphere of S_2. The congruences might also have been

$$2\mu x_3 - x_1 + ix_4 = 0, \quad 2\mu x_1 - 2\mu^2 x_3 + x_5 = 0.$$

The directrices in this case are

$$S_1' \equiv (0, -1, 0, 0, i, 2\mu),$$
$$S_2 \equiv (0, 2\mu, 0, 1, 0, -2\mu^2).$$

The ruled cyclide is the same in each case, and S_1' and S_1 have the same sphere in common with the cyclide. The spheres S_1 touch all the spheres S_1', i.e. the two pencils have a minimum line in common.

The complex can be generated in two ways by spheres which touch corresponding spheres of a tangent pencil and a cyclide projectively related. The cyclide and the two tangent pencils form the surface of singularities.

XI.

Eleventh Canonical Form. [6]

1. $$\Pi = x_1x_6 + x_2x_5 + x_3x_4 = 0,$$
$$F = 2a_1(x_1x_6 + x_2x_5 + x_3x_4) + 2x_1x_5 + 2x_2x_4 + x_3^2 = 0,$$
$$f = \frac{2(x_1x_6 + x_2x_5 + x_3x_4)}{k + a_1} - \frac{2x_1x_5 + 2x_2x_4 + x_3^2}{(k + a_1)^2} + \frac{2x_1x_4 + 2x_2x_3}{(k + a_1)^3}$$
$$- \frac{2x_1x_3 + x_2^2}{(k + a_1)^4} + \frac{2x_1x_2}{(k + a_1)^5} - \frac{x_1^2}{(k + a_1)^6} = 0.$$

It was seen that the surface [1122] becomes the surface [114] when the two double spheres become consecutive, in a similar manner the surface [222] becomes [24] when two of the double spheres become consecutive. Thus the surface [6] has three consecutive double spheres which have a minimum line in common; it is of order 12 and can be generated in one (special way as the envelope of ∞^2 spheres. The minimum line enveloped by $x_1 = x_2 = x_3 = 0$ is a part of the surface.

The Laguerre surface is of order 8, class 4.

XII.

The following table contains a complete list of those surfaces which appear as surfaces of singularities of the quadratic complex. The cyclide and spheroquartic are not included here, since they have been classified by Loria in the paper referred to.

The following symbols are used:

ν_1 = complex of planes.

Π = plane at infinity.

$(\bar{E})$ = Laguerre transform.

S_2 = central quadric.

$\bar{S}_2$ = paraboloid.

λ = complex of points.

quad. = skew quadrilateral of minimum lines.

C_4 = Dupin cyclide.

C_2 = cone of revolution.

NON-ANNULAR SURFACES.

General.	Order.	Class.	Laguerre.	Order.	Class.
111111. General surface,	24	24	No Laguerre surface.		
11112. One double sphere,	20	20	General Laguerre surface,	12	4
1113. One cuspidal sphere,	18	18	$(\bar{E})$ of $\bar{S}_2$,	10	4
114. One tacnodle sphere,	16	16	$(\bar{E})$ of $\bar{S}_2$, touching Π in point of K,	9	4
15. Singular sphere arises from union of double and cuspidal sphere,	14	14	$(\bar{E})$ of $\bar{S}_2$, touching Π in point of K; one generator in Π tangent to K,	8	4
1122. Two double spheres,	16	16	$(\bar{E})$ of S_2 touching K,	10	4
123. One double, one cuspidal sphere,	14	14	If $x_4 = \nu$, $(\bar{E})$ of $\bar{S}_2$, touching K,	8	4
			If $x_2 = \nu$, $(\bar{E})$ of S_2 having three point contact with K,	9	4
33. Two cuspidal spheres,	12	12	One cuspidal plane,	7	4

SURFACES WHICH CONTAIN A MINIMUM LINE AS PART OF THE ENVELOPE.

General.	Order.	Class.	Laguerre.	Order.	Class.
222. Three double spheres,	12	12	Two double planes,	8	4
24. One double, one tacnodal sphere	12	12			
			If $x_3 = \nu$, one double plane,	7	4
			If $x_1 = \nu$, one tacnodal plane,	8	4
6. Three coincident double spheres,	12	12		7	4

ANNULAR SURFACES.			DEVELOPABLES.		
(11) 1111. Two distinct double directrices,	16	16	$(\bar{E})$ of quadric cone,	8	4
111 (12). Coincident double directrices,	16	16	If $x_5 = \nu$, $(\bar{E})$ of cylinder,	8	4
			If $x_5 = \nu$, $x_4 = \lambda$, focal developable of central quadric,	8	4
11 (13). Single directrix which is also a double generator,	12	12	$(\bar{E})$ of parabolic cylinder,	6	4
			If $x_3 = \lambda$, focal developable of paraboloid,	6	4
1 (14). Single directrix which is cuspidal generator,	10	10	$(\bar{E})$ of parabolic cylinder in which line in Π touches K,	5	4
			If $x_2 = \lambda$ focal developable of $\bar{S}_2$ touching Π in point of K,	5	4
(11) 112. Two double directrices and double generator,	12	12	If $x_5 = \nu$, $(\bar{E})$ of $\bar{S}_2$ of revolution,*	8	4
			If $x_1 + ix_2 = \nu$, $(\bar{E})$ of cone with minimum plane,	6	4
(11) 13. Two double directrices and cuspidal generator,	10	10	If $x_4 = \nu$, $(\bar{E})$ of $\bar{S}_2$, of revolution,*	6	4
			If $x_1 + ix_2 = \nu$, $(\bar{E})$ of cone having three point contact with K,	5	4

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* These surfaces are annular.

General.	Order.	Class.	Laguerre.	Order.	Class.
12(12). Coincident double directrices and double generator,	12	12	If $x_5=\nu$, $(\bar{E})$ of cylinder which touches K,	6	4
			If $x_3=\nu$, $(\bar{E})$ of S_2 of revolution with minimum axis,*	8	4
			If $x_5=\nu$, $x_2=\lambda$, focal developable of S_2 which touches K,	6	4
(12)3. Coincident double directrices and cuspidal generator,	10	10	If $x_2=\nu$, $x_1=\lambda$, focal developable of S_2 having three point contact with K,	5	4
			If $x_2=\nu$, $(\bar{E})$ of above,*	5	4
			If $x_4=\nu$, $(\bar{E})$ of parabola which lies in minimum plane,*	6	4

FACTORABLE ANNULAR AND DEVELOPABLE SURFACES.

(11)22. One double, one single directrix,	8	8	If $x_5=\nu$, $(\bar{E})$ of conic passing through one circle point,	6	4
			If $x_1+ix_2=\nu$, developable circumscribed to sphere along cubic,	4	3
(11)4. One double, one single directrix,	8	8	If $x_3=\nu$, $(\bar{E})$ of parabola passing through a circle point,	5	4
			If $x_1=ix_2=\nu$, developable circumscribed to a sphere along a cubic,	4	3
2(13). Single directrix,	8	8	If $x_2=\nu$, annular,	6	3
			If $x_4=\nu$, developable,	4	3
			If $x_4=\nu$, $x_1=\lambda$, focal developable of S_2 tangent to K,	4	3

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* These surfaces are annular.

General.	Order.	Class.	Laguerre.	Order.	Class.
(15). Single directrix,	8	8	If $x_2 = \nu$, $x_1 = \lambda$, focal developable of $\bar{S}_2$ touching Π in point of K and one generator in Π tangent to K,	4	3
			If $x_2 = \nu$, $(\bar{E})$ of above,	4	3
Two C_4.			*Two C_2.*		
(11)(11)11. Two C_4 having two spheres of each generation in common.			Two C_2 having two common planes.		
1(11)(12). Two C_4 touching along a circle.			If $x_2 + ix_3 = \nu$, two C_2 tangent along an element.		
			If $x_5 = \nu$, two cylinders.		
			If $x_5 = \nu$, $x_4 = \lambda$, focal developable of S_2 of revolution, i. e. two minimum cones with foci of meridian section for vertices.		
(12)(12) Two C_4 touching along two circles.			If $x_5 = \nu$, two C_2 having an element in common.		
			If $x_5 = \nu$, $x_2 = \lambda$, focal developable of cylinder which touches K.		
11(22). Two ruled cyclides, the double line is double directrix.			Two cylinders with minimum axis.		
1(23). Two ruled cyclides, the double line is single directrix and single generator.			Two parabolic cylinders with minimum axis.		
C_4 and quad.			*C_2 and quad.*		
(11)(11)2. The quad belongs to two non-consecutive generators of C_4.			C_2 and two finite minimum lines and line at infinity in their plane counted twice.		
			If $x_5 = \nu$, C_2 and quadrilateral inscribed in K.		
(11)(13). The quad lies on one generator of C_4.			If $x_1 + ix_2 = \nu$, C_2 and two minimum lines lying in same tangent plane.		
			If $x_4 = \nu$, $x_3 = \lambda$, focal developable of the paraboloid of revolution.		
			If $x_4 = \nu$, C_2 and line in Π.		
2(22). A ruled cyclide: the quadrilateral becomes two lines and a line cutting them counted twice.			A cone with vertex on K.		
(24). Same as above			Same as above.		

C_4 Counted Twice.	*C_2 Counted Twice.*
(111) 111. The complex consists of spheres which meet paired spheres of C_4 in an involution [2].	The complex consists of those spheres which touch paired planes and of a cone of revolution in involution [2].
(111)(11) 1. The complex consists of spheres which touch paired spheres of a C in an involution [2] having two pairs of double elements coincident.	As above, the involution having two pairs of coincident elements.
(111)(111). Complex consists of spheres which touch a Dupin cyclide.	Complex consists of spheres which touch a cone of revolution.
(111)(12). Same as (111) 111 involution has two coincident double elements.	Same as (111) 111, involution has two coincident double elements.
(111)(12). As above, involution has all its double elements coincident.	As above, the involution has all its double elements coincident.
(111) 3. Involution has three double elements coincident.	Involution has three double elements coincident.
(222). Complex consists of those spheres which touch a ruled cyclide.	Complex consists of those spheres which touch a cylinder which has a minimum axis.

SURFACES WHICH DEGRADE INTO MINIMUM LINES.

(11)(11)(11). Two skew quadrilaterals.	One finite quadrilateral and quadrilateral inscribed in K.
11(112). A quadrilateral counted twice. Complex formed by spheres which meet corresponding spheres of two pencils in (2, 2) correspondence having a self-corresponding sphere.	If $x_5 = \nu$, quadrilateral inscribed in K.
	If $x_1 + ix_2 = \nu$, one of the lines becomes tangent to K.
1(113). As above, the (2, 2) correspondence has three coincident double elements.	As above.

(11)(112). As above, the pencils are in (1, 1) correspondence and have a self-corresponding sphere.	As above.
(112) 2. As above. The pencils are in (i, 2) correspondence.	As above.
1 (122). Two intersecting minimum lines each counted four times.	A finite minimum line and a line at infinity which cuts it, each counted four times.
(11)(22). Two skew quadrilaterals having two common sides.	Two minimum lines and one of their common secants counted twice.
(114). Same as (112) 2.	Same as (112) 2.
(123). Two intersecting minimum lines counted four times.	If $x_2 = \nu$, quadrilateral in K counted twice. If $x_4 = \nu$, two finite intersecting minimum lines counted twice and line at infinity in their plane counted four times.
(33). Two minimum lines counted three times and two counted once.	One minimum line counted three times and one counted once.

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